

On modelling bicycle power-meter measurements for velodromes

Motion of centre of mass for individual pursuits

Michael A. Slawinski*, Raphaël A. Slawinski†, Theodore Stanoev‡

June 11, 2020

Abstract

For a moving bicycle, the power meters respond to the propulsion of the centre of mass of the bicycle-cyclist system. Hence, an accurate modelling of power measurements—on a velodrome—requires a distinction between the trajectory of the wheels and the trajectory of the centre of mass. We formulate and examine an individual-pursuit model that takes into account the aforementioned distinction. In doing so, we provide the details of the invoked physical principles and mathematical derivations.

1 Introduction

In this article, we consider a mathematical model to account for power-meter measurements on velodromes. This work is a mathematization of certain aspects of studies presented by Martin et al (1998) and Underwood (2012). The discussed model is pertinent to individual pursuits, since we assume a cyclist to follow the black-line, in a constant aerodynamic position, with a constant black-line speed.

Using this model, for a given cyclist, we can calculate the power required to achieve a desired time or—since the relation between power and speed is one-to-one—the time achievable with a particular power. Also, for repeated laps, we can estimate model parameters from the power and speed measurements. The method proposed to estimate these parameters is specific to velodromes, and different from the circuit study presented by Chung (2012).

We begin this paper by presenting a model and justifying its mathematical formulation. We illustrate both forward and inverse applications of the model. We conclude this article by a discussion of results. This article contains also two appendices.

*Memorial University of Newfoundland, Canada, msslawins@mac.com

†Mount Royal University, Canada, rslawinski@mtroyal.ca

‡Memorial University of Newfoundland, Canada, theodore.stanoev@gmail.com

2 Formulation

A mathematical model to account for the power required to propel a bicycle—along a straight course—with speed V_{\rightarrow} is (e.g., Danek et al, 2020a)

$$\begin{aligned}
 P &= F_{\leftarrow} V_{\rightarrow} \\
 &= \frac{\overbrace{m g \sin \Theta}^{\text{gravity}} + \overbrace{m a}^{\text{change of speed}} + \overbrace{C_{\text{rr}} m g \cos \Theta}^{\text{rolling resistance}} + \overbrace{\frac{1}{2} \eta C_{\text{d}} A \rho (V_{\rightarrow} + w_{\leftarrow})^2}^{\text{air resistance}}}{\underbrace{1 - \lambda}_{\text{drivetrain efficiency}}} V_{\rightarrow},
 \end{aligned} \tag{1}$$

where F_{\leftarrow} stands for the forces opposing the motion and V_{\rightarrow} for the ground speed. In particular, m is the mass of the cyclist and the bicycle, g is the acceleration due to gravity, Θ is the slope of a hill, a is the change of speed, C_{rr} is the rolling-resistance coefficient, $C_{\text{d}}A$ is the air-resistance coefficient, ρ is the air density, w_{\leftarrow} is the wind component opposing the motion, λ is the drivetrain-resistance coefficient, η is a quantity that ensures the proper sign for the tailwind effect, $w_{\leftarrow} < -V_{\rightarrow} \iff \eta = -1$, otherwise, $\eta = 1$.

To consider a steady ride, $a = 0$, on a flat course, $\Theta = 0$, in windless conditions, $w = 0$, we write

$$P = \underbrace{\frac{C_{\text{rr}} m g + \frac{1}{2} C_{\text{d}} A \rho V_{\rightarrow}^2}{1 - \lambda}}_{F_{\leftarrow}} V_{\rightarrow}. \tag{2}$$

In modelling the power along a straight course, there is no distinction between the ground speed of the centre of mass and of any other point of the bicycle-cyclist system. The distinction appears if the cyclist deviates from a straight course by leaning, which becomes pronounced on a velodrome.

Let us consider a velodrome whose black-line distance is C , and the banking angle and radius are θ and r , respectively. Also, let us assume that the centre of mass of the bicycle-cyclist system is h above the ground, without lean.¹ If so, the radius of the centre-of-mass trajectory, r_{CoM} , is shorter than r by $h \sin \vartheta$, where ϑ is the angle, measured from the vertical, at which the cyclist leans. Hence, the distance traveled—in one lap—by the centre of mass is shorter than the black line by

$$2\pi r - 2\pi \overbrace{(r - h \sin \vartheta)}^{r_{\text{CoM}}} = 2\pi h \sin \vartheta. \tag{3}$$

Thus—neglecting a progressive leaning and straightening between the straights and the banks—the distance travelled by the centre of mass, along the straights and along the banks, for a single lap, is

$$C - 2\pi r \quad \text{and} \quad 2\pi(r - h \sin \vartheta), \tag{4}$$

respectively.

Given a laptime,

$$t_{\odot} = \frac{C}{V_{\rightarrow}}, \tag{5}$$

¹Herein, we assume that the position of a cyclist on a bicycle remains the same, which is a reasonable assumption for an individual pursuit, after the initial acceleration. Hence, h is constant, and the change of the height of the centre of mass is due only to the lean angle, ϑ .



Figure 1: Christiaan Huygens confident of a centripetal force, which he mathematized in 1659

we obtain \bar{V}_{\rightarrow} , which is the average black-line speed, per lap. For the centre of mass, under the assumption of a constant black-line speed,

$$t_{\odot} = \frac{C - 2\pi r}{\bar{V}_{\rightarrow}} + \frac{2\pi(r - h \sin \vartheta)}{\bar{V}_{\curvearrowright}}, \quad (6)$$

where \bar{V}_{\rightarrow} is the average centre-of-mass speed along the straights and $\bar{V}_{\curvearrowright}$ is the average centre-of-mass speed along the banks. \bar{V}_{\rightarrow} is both the black-line speed and the centre-of-mass speed along the straights, since the lap average of a constant speed is the same as the average for any lap segment, and — along the straights — there is no distinction between the speed of the centre of mass and the speed of any point of the bicycle-cyclist system.

The lean angle of a cyclist — as illustrated in Figures 1 and 2, and entailed by expression (18), below, as well as discussed in Appendix A — is

$$\vartheta = \arctan \frac{F_{cp}}{F_g}, \quad (7)$$

where the magnitude of the centripetal force is

$$F_{cp} = \frac{m V_{\curvearrowright}^2}{r_{\text{CoM}}}, \quad (8)$$

and of the force of gravity is $F_g = m g$; hence,

$$\bar{V}_{\curvearrowright} = \sqrt{g(r - h \sin \vartheta) \tan \vartheta}. \quad (9)$$

Inserting expression (9) into expression (6), we obtain an equation for ϑ , whose solution entails $\bar{V}_{\curvearrowright}$.

Following expressions (4), the proportion of distance travelled, per lap, by the centre of mass, is

$$1 - \frac{2\pi r}{C} \quad \text{and} \quad \frac{2\pi(r - h \sin \vartheta)}{C},$$

along the straight and the bank, respectively. Assuming a constant black-line speed, the proportion of the riding time along the straights and the banks is the same as the proportion of their distances. Thus — according to the harmonic average, discussed in Appendix B — the average centre-of-mass speed, per lap, is

$$\langle V \rangle = \frac{1}{\frac{1}{\bar{V}_{\rightarrow}} \left(\frac{1 - \frac{2\pi r}{C}}{1 - \frac{2\pi h \sin \vartheta}{C}} \right) + \frac{1}{\bar{V}_{\curvearrowright}} \left(\frac{\frac{2\pi(r - h \sin \vartheta)}{C}}{1 - \frac{2\pi h \sin \vartheta}{C}} \right)} \quad (10a)$$

$$= \frac{\bar{V}_{\rightarrow} \bar{V}_{\curvearrowright} (C - 2\pi h \sin \vartheta)}{C \bar{V}_{\curvearrowright} + 2\pi (r (\bar{V}_{\rightarrow} - \bar{V}_{\curvearrowright}) - \bar{V}_{\rightarrow} h \sin \vartheta)}. \quad (10b)$$

The average power per lap is

$$\bar{P} = \frac{1}{1 - \lambda} \underbrace{\frac{\bar{V}_{\rightarrow} \bar{V}_{\curvearrowright} (C - 2\pi h \sin \vartheta)}{C \bar{V}_{\curvearrowright} + 2\pi (r (\bar{V}_{\rightarrow} - \bar{V}_{\curvearrowright}) - \bar{V}_{\rightarrow} h \sin \vartheta)}}_{\langle V \rangle} \left\{ \quad (11a)$$

$$C_{\text{rr}} m g \left(1 - \frac{2\pi r}{C} \right) \quad (11b)$$

$$+ \left(C_{\text{rr}} \underbrace{m g (\sin \theta \tan \vartheta + \cos \theta)}_N \cos \theta + C_{\text{sr}} m g \underbrace{\frac{\sin(\theta - \vartheta)}{\cos \vartheta}}_{F_f} \sin \theta \right) \frac{2\pi \overbrace{(r - h \sin \vartheta)}^{r_{\text{CoM}}}}{C} \quad (11c)$$

$$+ \frac{1}{2} C_{\text{d}} A \rho \left(\left(1 - \frac{2\pi r}{C} \right) \bar{V}_{\rightarrow}^2 + \frac{2\pi \overbrace{(r - h \sin \vartheta)}^{r_{\text{CoM}}}}{C} \bar{V}_{\curvearrowright}^2 \right) \left. \right\}, \quad (11d)$$

with ϑ , \bar{V}_{\rightarrow} and $\bar{V}_{\curvearrowright}$ that result from equations (5), (6) and (9). Herein, C_{sr} is the coefficient of the lateral friction. If $C \rightarrow \infty$, expression (11) reduces to expression (2), as expected.

The second fraction in factor (11a) is the average centre-of-mass speed, per lap, which combines, proportionally, \bar{V}_{\rightarrow} , along the straights, and $\bar{V}_{\curvearrowright}$, along the banks. Summand (11b) is the rolling resistance along the straights.² Summand (11d) is the air resistance, which is a function of proportions between \bar{V}_{\rightarrow}^2 , along the straights, and $\bar{V}_{\curvearrowright}^2$, along the banks.³

²A velodrome, such as Vélodrome de Bordeaux-Lac, has $C = 250$, $r = 23$, $\theta = \pi/4$, along the banks, $\Theta = 0.23$, along the straights; the latter is not zero to avoid an excessive change of track inclination between the banks and the straights. We could refine the model by including,

$$(C_{\text{rr}} \cos^2 \Theta + C_{\text{sr}} \sin^2 \Theta) m g \left(1 - \frac{2\pi r}{C} \right),$$

to accounts for the force of gravity along the straights, which corresponds to summand (11c), with $\vartheta = 0$. However, since Θ is small, $\cos \Theta \approx 1$ and $\sin \Theta \approx 0$, it suffices to consider

$$C_{\text{rr}} m g \left(1 - \frac{2\pi r}{C} \right),$$

which is summand (11b).

³We could refine the model by including the effect of air resistance of rotating wheels (Danek et al, 2020a, Appendix D), which would require introducing another resistance coefficient to summand (11d), if the wheels are the same, or two coefficients, if they are different.

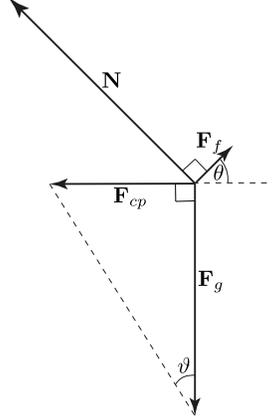


Figure 2: Force diagram: Inertial frame

To formulate summand (11c), we use the relations among the magnitudes of vectors \mathbf{N} , \mathbf{F}_g , \mathbf{F}_{cp} and \mathbf{F}_f , illustrated in Figure 2. In accordance with Newton's second law, for a cyclist to maintain a horizontal trajectory, the resultant of all vertical forces must be zero,

$$\sum F_y = 0 = N \cos \theta + F_f \sin \theta - F_g. \quad (12)$$

In other words, \mathbf{F}_g must be balanced by the sum of the vertical components of normal force, \mathbf{N} , and the friction force, \mathbf{F}_f , which is parallel to the velodrome surface and perpendicular to the instantaneous velocity. Depending on the centre-of-mass speed and on the radius of curvature for the centre-of-mass trajectory, if $\vartheta < \theta$, \mathbf{F}_f points upwards, in Figure 2, which corresponds to its pointing outwards, on the velodrome; if $\vartheta > \theta$, it points downwards and inwards. If $\vartheta = \theta$, $\mathbf{F}_f = \mathbf{0}$. Since we assume no lateral motion, \mathbf{F}_f accounts for the force that prevents it. Heuristically, it can be conceptualized as the force exerted in a lateral deformation of the tires.

For a cyclist to follow the curved bank, the resultant of the horizontal forces,

$$\sum F_x = -N \sin \theta + F_f \cos \theta = -F_{cp}, \quad (13)$$

is the centripetal force, \mathbf{F}_{cp} , whose direction is perpendicular to the direction of motion and points towards the centre of the radius of curvature. According to the rotational equilibrium about the centre of mass,

$$\sum \tau_z = 0 = F_f h \cos(\theta - \vartheta) - N h \sin(\theta - \vartheta), \quad (14)$$

where τ_z is the torque about the axis parallel to the instantaneous velocity, which implies

$$F_f = N \tan(\theta - \vartheta). \quad (15)$$

Substituting expression (15) in expression (12), we obtain

$$N = \frac{m g}{\cos \theta - \tan(\theta - \vartheta) \sin \theta} = m g (\sin \theta \tan \vartheta + \cos \theta). \quad (16)$$

Using this result in expression (15), we obtain

$$F_f = m g (\sin \theta \tan \vartheta + \cos \theta) \tan(\theta - \vartheta) = m g \frac{\sin(\theta - \vartheta)}{\cos \vartheta}. \quad (17)$$

To relate F_{cp} and ϑ , we use these results in expression (13), to obtain

$$F_{cp} = N \sin \theta - F_f \cos \theta = m g \tan \vartheta, \quad (18)$$

which is tantamount to expression (7). Examining expressions (8) and (18), we see that the lean angle is a function of the centre-of-mass speed and of the radius of curvature for the centre-of-mass trajectory; it is independent of mass or the track inclination.

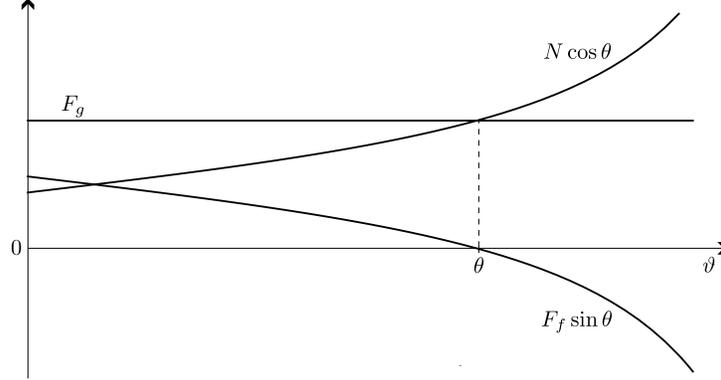


Figure 3: F_g in terms of vertical components of \mathbf{N} and \mathbf{F}_f

In terms of solutions (16) and (17), expression (12)—as a function of ϑ , for a fixed value of θ —is shown in Figure 3. F_g is constant, as required. Also, as required, $F_f = 0$ and $N \cos \theta = F_g$, at $\vartheta = \theta$. For $\vartheta < \theta$, \mathbf{F}_f points outwards, hence—in accordance with Figure 2—it is positive. For $\vartheta > \theta$, \mathbf{F}_f points inwards and, hence, is negative. The crossing of two curves corresponds to ϑ at which the vertical components of \mathbf{N} and \mathbf{F}_f are equal to one another.

3 Numerical example

3.1 Forward model

⁴Let us consider the following values. For the bicycle-cyclist system, $m = 111$, $h = 1.2$, $C_d A = 0.2$, $C_{rr} = 0.002$, $C_{sr} = 0.003$ and $\lambda = 0.02$. For the velodrome, $C = 250$, $r = 23$ and $\theta = 0.7505$. For the external conditions, $g = 9.81$ and $\rho = 1.225$.

Let the laptime be such that, according to expression (5), the corresponding black-line speed is $\bar{V}_{\rightarrow} = 12$. In accordance with expressions (6) and (9), $\vartheta = 0.555468$, and $\bar{V}_{\curvearrowleft} = 11.6698$. Hence, in accordance with expression (11), $\bar{P} = 229.6723$. Also, in accordance with expression (10), the average centre-of-mass speed, per lap, is $\bar{V}_{\curvearrowleft} < \langle V \rangle = 11.8091 < \bar{V}_{\rightarrow}$, as expected.

The values of summands (11b), (11c) and (11d), which represent the forces opposing the movement, are, respectively, 0.9189, 1.3195 and 25.2319. The air resistance, which is the third summand, has the dominant effect, with respect to the other summands, which are associated with the effect of the wheel contact with the velodrome surface.

If the centre of mass is not taken into account, expression (2) results in $\bar{P} = 243.3458$, which is an overestimate. The values of the first and second summands in the numerator of expression (2) are

⁴For consistency with power meters, whose measurements are expressed in watts, which are $\text{kg m}^2/\text{s}^3$, we use the *SI* units for all quantities. Mass is given in kilograms, kg, length in meters, m, and time in seconds, s; hence, speed is in m/s; angles are in radians.

2.1778 and 26.4600, respectively. Therein, the first summand corresponds to the effect of the wheel contact with the surface and the second to the effect of the air resistance.

The similarity between the values obtained with expressions (2) and (11) is a supportive evidence for the correctness of refinements provided by the latter and an indication of a reasonable accurateness of the former, in spite of its simplicity. However, recognizing the difference between these values is consistent with the attention to marginal gains that underpin the Team GB dominance in velodrome competitions (e.g., Slater, 2012).

For this numerical example, $\mathbf{N} = [-857.4810, 919.5203]$, $\mathbf{F}_g = [0, -1088.91]$, $\mathbf{F}_{cp} = [-675.8359, 0]$ and $\mathbf{F}_f = [181.6451, 169.3897]$. These forces are illustrated in Figure 2; the positive directions are upward and rightward. As required, the resultant of the vertical forces is zero, and the resultant of the horizontal forces is equivalent to \mathbf{F}_{cp} . Also, the orientation of \mathbf{F}_f is $\theta = 0.7505$, which means that it is parallel to the velodrome surface, as required.

3.2 Inverse problem

For this numerical example, expression (11) can be written as

$$229.6723 = 993.2258 \underbrace{\frac{C_d A}{1-\lambda}}_X + 11530.0934 \underbrace{\frac{C_{rr}}{1-\lambda}}_Y + 1124.4934 \underbrace{\frac{C_{sr}}{1-\lambda}}_Z. \quad (19)$$

In contrast to expression (1), for expression (2), and for its extension, given by expression (11), the resistance coefficients appear only as ratios. Hence, even with many independent equations, we cannot obtain — as an inverse solution — the values of $C_d A$, C_{rr} , C_{sr} and λ , but only the ratios, X , Y and Z .

To obtain the values of X , Y and Z , we perform the least-squares fit of ten equations analogous to equation (19), with $\vec{V}_{\rightarrow} \in (11.5, 12.5)$, whose matrix representation is

$$\begin{bmatrix} 250.2814 \\ 221.2244 \\ 246.4302 \\ 207.1293 \\ 210.2785 \\ 234.5504 \\ 231.1933 \\ 225.5622 \\ 221.3932 \\ 235.9688 \end{bmatrix} = \begin{bmatrix} 1090.8398 & 12040.9534 & 1008.6335 \\ 953.2952 & 11315.5242 & 1169.9259 \\ 1072.5787 & 11946.7665 & 1030.7921 \\ 886.7918 & 10950.0690 & 1242.7456 \\ 901.6366 & 11032.5717 & 1226.8169 \\ 1016.3057 & 11652.5725 & 1097.6921 \\ 1000.4207 & 11568.3931 & 1116.1798 \\ 973.7925 & 11426.0999 & 1146.7564 \\ 954.0928 & 11319.8441 & 1169.0305 \\ 1023.0198 & 11687.9987 & 1089.8239 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \quad (20)$$

The least-squares solution of system (20) is $X = 0.204082$, $Y = 0.002041$, $Z = 0.003061$. Since $\lambda = 0.02$, we obtain $C_d A = 0.2$, $C_{rr} = 0.002$, $C_{sr} = 0.003$, as expected.

For the measured values of P — as opposed to the modelled ones, for which three equations suffice to solve for X , Y and Z — a redundancy of the laptime information allows us to estimate them, and to obtain statistical information about the empirical adequacy of a model, which, however sophisticated, remains only a mathematical analogy for a physical realm. This redundancy could correspond to different laptimes during a single ride.

An insight into the consistency of information can be gained by writing each equation of system (20)

as

$$1 = \frac{a}{\bar{P}} \underbrace{\frac{C_d A}{1-\lambda}}_X + \frac{b}{\bar{P}} \underbrace{\frac{C_{rr}}{1-\lambda}}_Y + \frac{c}{\bar{P}} \underbrace{\frac{C_{sr}}{1-\lambda}}_Z,$$

and plotting a/\bar{P} , b/\bar{P} and c/\bar{P} . For system (20), they are collinear. For measurements, the departure from the collinearity is indicative of the quality of the model and of the measurement errors. Within a model, such a plot can be used to study the sensitivity of X , Y and Z to perturbations.

To estimate $C_d A$, C_{rr} and C_{sr} , the value of λ needs to be given independently or be assumed. It is commonly accepted that, for high-quality track bicycles, $\lambda \in (0.01, 0.03)$. Also, if the power meter is in the rear hub, as opposed to being in the pedals or the bottom bracket, $\lambda \approx 0$, since the effect of the resistance of the drivetrain—which includes the chainring, chain and sprocket—upon the measuring device is nearly eliminated (e.g., Chung, 2012).

4 Conclusions

Expression (11), together with expressions (5), (6) and (9), allows us to calculate the power required to achieve a desired individual-pursuit time or the time achievable with a particular power. It also allows us to quantify the effects of the bicycle-cyclist weight, air resistance, rolling resistance, drivetrain resistance and lateral friction, as well as of the velodrome size, steepness of its banks and tightness of its curves. Furthermore, a quantification of these effects lends itself to a study of optimization of a cyclist’s effort (Danek et al, 2020b, Appendix A).

Our formulation and examination of an inverse solution for expression (11) shows that we can infer only the ratios, $C_d A/(1-\lambda)$, $C_{rr}/(1-\lambda)$ and $C_{sr}/(1-\lambda)$. Nevertheless, the proposed solution allows us to gain an insight into the consistency between the measurements and the model.

In view of presented results, we conclude that accurate inferences based on the power-meter measurements on a velodrome require a distinction between the trajectory of wheels, which herein we assume to coincide with the black line, and the trajectory of the centre of mass. The forces involved and relations among them, as well as other entailed quantities, are functions of the latter.

If necessary, to model highly accurate measurements or for a specific scope of investigation, the mathematical model stated in expression (11) can be refined in a manner suggested in footnotes 2 and 3. In contrast to these refinements, a progressive leaning and straightening—neglected to formulate distances in expression (4), and resulting in two constant centre-of-mass speeds, \bar{V}_\rightarrow and \bar{V}_\curvearrowright , in expression (6)—cannot be achieved within this model, as discussed in Appendix C.

The results presented in this article might be an *a posteriori* reassurance and comfort for Michael Hutchinson (2006, p. 251), in his attempt to achieve immortality the hard way,

Ride fast—in the end that’s all it ever comes down to. The pressure of another curve, the relief of the simple straight. But the straight’s short respite is never enough. My shoulders are awful. My arms hurt. And every few seconds I have to manage another banked curve. Each one demands a little more effort, a little more concentration. As the physical toll mounts, the balance and rhythm aren’t offering the protection from reality that they did. I need some sort of reassurance. Some comfort.

The power required to maintain the same speed—with respect to the black line—is less on a curve than on a straight. In a certain manner, the curves provide a short respite. Also, thanks to the curves, the distance travelled by a centre of mass, within a given time, is shorter than with respect to the black line.

A Lean angle

Expressions (A.1) and (18) imply that the lean angle, ϑ , depends only on the centripetal acceleration of the cyclist's centre of mass and the acceleration of gravity. In other words, the lean angle depends only on the centre-of-mass speed and the radius of curvature of the centre-of-mass trajectory, not on the track inclination, θ , even though both the normal force, stated in expression (16), and the frictional force, in expression (17), do depend on track inclination. However, the θ -dependence cancels out of the centripetal force, stated in expression (18), by a seldom used trigonometric identity.

Given the generality of this result, it would be satisfying to obtain it in a manner that explains it in the context of physics. To this end, let us analyze the situation from inside the noninertial frame of the cyclist. Specifically, we consider the frame comoving with the cyclist around a curve, with the cyclist considered as a point mass. We neglect the additional accelerated motion resulting from the rotation of the cyclist about an axis through the centre of mass.

As illustrated in Figure A1 and in contrast to Figure 2, in this noninertial frame, instead of the forces in the horizontal direction summing to the centripetal force, \mathbf{F}_{cp} , the forces in this direction, including the fictitious centrifugal force, $\mathbf{F}_{cf} = -\mathbf{F}_{cp}$, sum to zero. The centrifugal force must be taken to act at the centre of mass, since otherwise the torque about the centre of mass, stated in expression (14), would be affected.

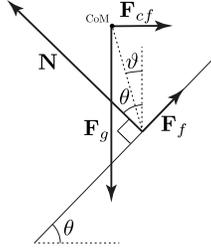


Figure A1: Force diagram: Noninertial frame

To proceed, we invoke the vector identity,

$$\sum \boldsymbol{\tau} = \mathbf{R}_{\text{CoM}} \times \sum \mathbf{F} + \sum \boldsymbol{\tau}_{\text{CoM}},$$

where $\sum \boldsymbol{\tau}$ is the net torque about an arbitrary point, $\sum \boldsymbol{\tau}_{\text{CoM}}$ is the net torque about the centre of mass, \mathbf{R}_{CoM} is the position vector of the centre of mass, and $\sum \mathbf{F}$ is the net force. From this identity there follows the well-known result that if the net force is zero and the net torque about the centre of mass is zero, the net torque about any other point is also zero. In particular, let us consider the torque about the point of contact of the tires with the surface,

$$\sum \tau_z = h F_g \sin \vartheta - h F_{cf} \sin \left(\frac{\pi}{2} - \vartheta \right) = 0,$$

from which — considering the magnitudes — it follows that

$$\begin{aligned} \tan \vartheta &= \frac{F_{cf}}{F_g} \\ &= \frac{F_{cp}}{F_g}, \end{aligned} \tag{A.1}$$

where expression (A.1) is equivalent to expressions (7) and (18). Expression (A.1) manifestly holds whether the curve is banked or unbanked.

Physically, from inside the noninertial frame of the cyclist, the value of the lean angle that obtains for specific values of speed and radius is the one that makes the gravitational torque balance the centrifugal torque. In particular, this condition makes no reference to track inclination.

B Harmonic mean

To calculate the average power over a lap, we require total work and time,

$$W = \oint F(s) ds \quad \text{and} \quad T = \oint dt = \oint \frac{ds}{V(s)},$$

to write

$$\bar{P} = \frac{W}{T} = \frac{\oint F(s) ds}{\oint \frac{ds}{V(s)}}.$$

If we consider n segments along which F and V are constant, we write

$$\bar{P} = \frac{\sum_{i=1}^n F_i \Delta s_i}{\sum_{i=1}^n \frac{\Delta s_i}{V_i}} \frac{1}{\Delta s} = \frac{\sum_{i=1}^n F_i \frac{\Delta s_i}{\Delta s}}{\sum_{i=1}^n \frac{1}{V_i} \frac{\Delta s_i}{\Delta s}}.$$

Since in our case, there are only two segments, we write

$$\bar{P} = \frac{F_{\rightarrow} \frac{\Delta s_{\rightarrow}}{\Delta s} + F_{\curvearrowright} \frac{\Delta s_{\curvearrowright}}{\Delta s}}{\frac{1}{V_{\rightarrow}} \frac{\Delta s_{\rightarrow}}{\Delta s} + \frac{1}{V_{\curvearrowright}} \frac{\Delta s_{\curvearrowright}}{\Delta s}}, \quad (\text{B.1})$$

where \rightarrow denotes straights and \curvearrowright denotes banks. The numerator is the distance-weighted arithmetic mean, \bar{F} . The denominator is reciprocal of the distance-weighted harmonic mean,

$$\langle V \rangle = \frac{\sum_{i=1}^2 \frac{\Delta s_i}{\Delta s}}{\sum_{i=1}^2 \frac{1}{V_i} \frac{\Delta s_i}{\Delta s}} = \frac{1}{\frac{1}{V_{\rightarrow}} \frac{\Delta s_{\rightarrow}}{\Delta s} + \frac{1}{V_{\curvearrowright}} \frac{\Delta s_{\curvearrowright}}{\Delta s}},$$

which — for the average centre-of-mass speed — results in expression (10a).

In contrast to $\langle V \rangle$, the distance-weighted arithmetic mean is

$$\bar{V} = \frac{\sum_{i=1}^2 V_i \Delta s_i}{\Delta s} = V_{\rightarrow} \frac{\Delta s_{\rightarrow}}{\Delta s} + V_{\curvearrowright} \frac{\Delta s_{\curvearrowright}}{\Delta s}, \quad (\text{B.2})$$

which, in view of expression (B.1), confirms that $\bar{P} \neq \bar{F} \bar{V}$. The harmonic mean is less than the arithmetic mean; it is skewed toward slower speeds. Thus, $\bar{F} \bar{V}$ would overestimate the average power.

For our numerical example, following expression (10), $\langle V \rangle = 11.8091$; hence, in accordance with expression (11), $\overline{F} \langle V \rangle = 229.6723$. On the other hand, following expression (B.2), $\overline{V} = 11.8114$, and $\overline{F} \overline{V} = 229.7161$. The difference is small due to the similarity of values of $\overline{V}_{\rightarrow}$ and $\overline{V}_{\curvearrowright}$. It would not be so, for an average of an upwind and downwind segments, discussed by Danek et al (2020b, Appendix A), where also the harmonic mean is used.

C Transition between curves and straights

In this article, to consider the velodrome in question, we assume that, along the banks, the radius of curvature is constant. Hence, the track is composed of two semicircles and two straights; this is the case of the light grey oval in Figure C1. The dark grey and black ovals also represent a track whose $C = 250$, but their radii of curvature are not constant; they are $r = 23$, at the intercepts with the horizontal axis, and $r \rightarrow \infty$, at the intercepts with the vertical axis. The dark grey oval is composed of an ellipse (Benham et al, 2020), whose semiaxes are 23 and 30, and of two straights. The black oval, in polar coordinates, is

$$r(\phi) = 28.15 (1 + 0.5 \sin^2 \phi \cos^2 \phi + 0.7 \cos^4 \phi), \quad \phi = [0, 2\pi), \quad (\text{C.1})$$

where the coefficients are found numerically by invoking the concept of the arclength and curvature to ensure that $C = 250$, $r_{\min} = 23$, $r_{\max} \rightarrow \infty$.

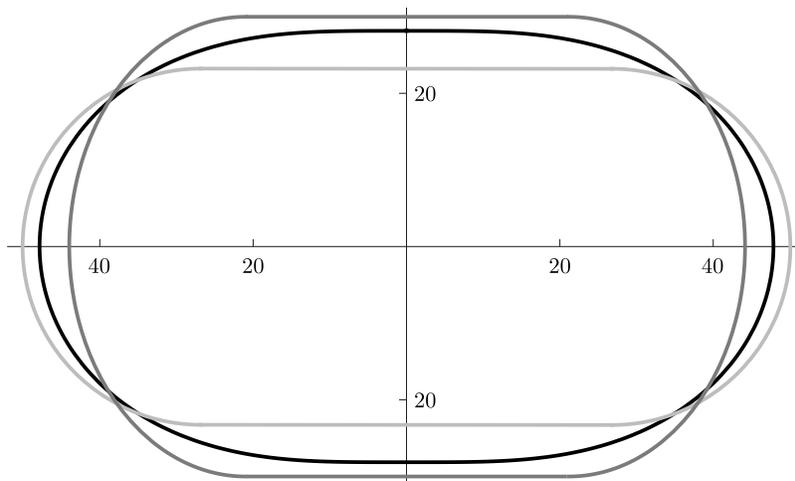


Figure C1: Velodrome tracks

These ovals share important geometrical properties, namely, their circumference and their radii of curvature, at the horizontal-axis and vertical-axis intercepts. However, the model presented in this article applies explicitly to the light grey oval. Its application to other similar ovals entails a decrease in accuracy.

For the light-grey oval, a model requires two constant centre-of-mass speeds, for the straights and for the circular banks. For the dark-grey oval, an explicit model requires the centre-of-mass speed to be represented by two functions, where the speed along the straight is constant, but along the banks is not, due to the changing radius of curvature along the elliptical bank. For the black oval, an explicit model requires the centre-of-mass speed to be represented by a single function, which depends on the continuously changing radius of curvature.

Let us examine the black-oval model. Its average curvature, for the length of a lap, is

$$\begin{aligned} \bar{\kappa} &= \frac{\int_0^C \kappa(s) ds}{\int_0^C ds} = \frac{\int_0^{2\pi} \frac{\left| r^2(\phi) + 2 \left(\frac{\partial r(\phi)}{\partial \phi} \right)^2 - r(\phi) \frac{\partial^2 r(\phi)}{\partial \phi^2} \right|}{\left(\sqrt{r^2(\phi) + \left(\frac{\partial r(\phi)}{\partial \phi} \right)^2} \right)^3} \sqrt{r^2(\phi) + \left(\frac{\partial r(\phi)}{\partial \phi} \right)^2} d\phi}{\int_0^{2\pi} \sqrt{r^2(\phi) + \left(\frac{\partial r(\phi)}{\partial \phi} \right)^2} d\phi} \\ &= \frac{\int_0^{2\pi} \frac{\left| r^2(\phi) + 2 \left(\frac{\partial r(\phi)}{\partial \phi} \right)^2 - r(\phi) \frac{\partial^2 r(\phi)}{\partial \phi^2} \right|}{r^2(\phi) + \left(\frac{\partial r(\phi)}{\partial \phi} \right)^2} d\phi}{\int_0^{2\pi} \sqrt{r^2(\phi) + \left(\frac{\partial r(\phi)}{\partial \phi} \right)^2} d\phi}; \end{aligned}$$

a numerical integration, with r given in expression (C.1), results in $\bar{\kappa} = 0.0251257$. Hence, the average radius of curvature is $\bar{r} := 1/\bar{\kappa} = 39.7998$. This is consistent with an expectation in view of the light-grey oval, whose radius of curvature is $r = 23$, along the banks, and infinity along the straights. For the black-oval model, in a manner analogous to expressions (3) and (4), the distance traveled — in one lap — by the centre of mass is

$$\begin{aligned} &\int_0^{2\pi} \sqrt{\left(r(\phi) - h \sin \bar{\vartheta} \right)^2 + \left(\frac{\partial \left(r(\phi) - h \sin \bar{\vartheta} \right)}{\partial \phi} \right)^2} d\phi \\ &= \int_0^{2\pi} \sqrt{\left(r(\phi) - h \sin \bar{\vartheta} \right)^2 + \left(\frac{\partial r(\phi)}{\partial \phi} \right)^2} d\phi, \end{aligned}$$

where $\bar{\vartheta}$ is the average lean angle. Thus, given a laptime, in a manner analogous to expression (6), we write

$$t_{\odot} = \frac{\int_0^{2\pi} \sqrt{\left(r(\phi) - h \sin \bar{\vartheta} \right)^2 + \left(\frac{\partial r(\phi)}{\partial \phi} \right)^2} d\phi}{\bar{V}}, \quad (\text{C.2})$$

where \bar{V} is the average speed, which — following expression (9) — we write as

$$\bar{V} = \sqrt{g \bar{r} \tan \bar{\vartheta}}. \quad (\text{C.3})$$

To compare the resulting power with the numerical example in Section 3, we let the laptime be such that, according to expression (5), the corresponding black-line speed is $\bar{V}_{\rightarrow} = 12$. In accordance with expressions (C.2) and (C.3), and as shown in Figure C2, we obtain $\bar{\vartheta} = 0.347284$, numerically, which results in $\bar{V} = 11.8878$. Hence, in accordance with expression (2), $\bar{P} = 236.415$.⁵ If we

⁵We could refine the black-oval model by including the effect of the track inclination, which — given a minimum and maximum values of inclination along the oval, stated in expression (C.1), as well as an interpolation formula between them — is illustrated in Figure C3. Hence, this effect could be expressed as a continuous function of distance.

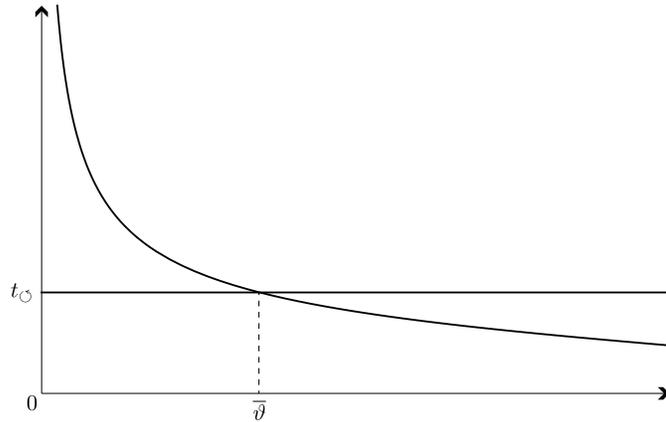


Figure C2: Left-hand and right-hand sides of equation (C.2)

consider the average centre-of-mass speed, per lap, for the light-grey oval, which—in accordance with expression (10)—is $\bar{V} = 11.8114$, we obtain $\bar{P} = 232.223$. Another comparison is the distance travelled by the centre of mass. For the black oval, it is 247.662; for the light-grey oval, it is 246.024.

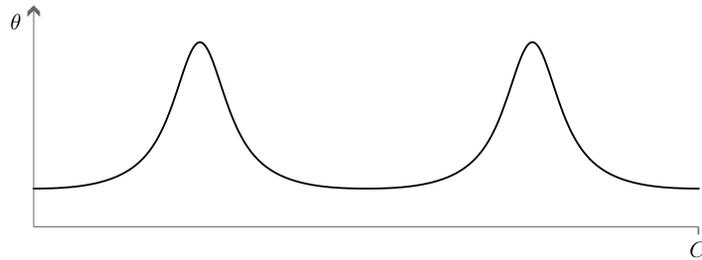


Figure C3: Track inclination

The model based on the light-grey oval requires fewer approximations, within the realm of mathematics. This is a consequence of the idealization of a velodrome track, which is greater for the light-grey oval than for the black oval, due to the assumption of a constant curvature and no transition between the curves and the straights. Thus, in spite of more mathematical approximations, the latter might exhibit a superior empirical adequacy. A further examination of this question, which is essential to the concept of modelling, requires experimental results. The conclusiveness of such results, however, might also be questionable, in view of the similarity of $\bar{P} = 236.35$, for the black oval, and $\bar{P} = 229.7161$, for the light-grey oval.

Acknowledgements

We wish to acknowledge Len Bos for his mathematical insights, Roger Mason for his perceptive comments, David Dalton, for his scientific editing and proofreading, Elena Patarini, for her graphic support, and Roberto Lauciello, for his artistic contribution. Furthermore, we wish to acknowledge Favero Electronics for inspiring this study by their technological advances and for supporting this work by providing us with their latest model of Assioma Duo power meters.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

- Benham GP, Cohen C, Brunet E, Clanet C (2020) Brachistochrone on a velodrome. arXiv: 1908.02224v2 [physics.flu-dyn]
- Chung R (2012) Estimating CdA with a power meter. URL <http://anonymous.coward.free.fr/wattage/cda/indirect-cda.pdf>
- Danek T, Slawinski MA, Stanoev T (2020a) On modelling bicycle power-meter measurements: Part I. Estimating effects of air, rolling and drivetrain resistance. arXiv: 2005.04229 [physics.pop-ph]
- Danek T, Slawinski MA, Stanoev T (2020b) On modelling bicycle power-meter measurements: Part II. Relations between rates of change of model quantities. arXiv: 2005.04480 [physics.pop-ph]
- Hutchinson M (2006) *The Hour: Sporting immortality the hard way*. Random House
- Martin JC, Milliken DL, Cobb JE, McFadden KL, Coggan AR (1998) Validation of a mathematical model for road cycling power. *Journal of Applied Biomechanics* 14:276–291
- Slater M (2012) Olympics cycling: Marginal gains underpin Team GB dominance. URL <https://www.bbc.com/sport/olympics/19174302>
- Underwood L (2012) *Aerodynamics of track cycling*. PhD thesis, University of Canterbury