

On maximizing VAM for a given power output

Slope, cadence, force and gear-ratio considerations

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Abstract

For a constant power output, the mean ascent speed (VAM) increases monotonically with the slope. Also, to maximize the ascent speed, the slope needs to be constant. These properties constitute a mathematical-physics background upon which various strategies for the VAM maximization can be examined in the context of the maximum sustainable power as a function of both the gear ratio and cadence.

1 Introduction

In this paper, we consider a mathematical model that accounts for power-meter measurements. In accordance with this model, VAM (*velocità ascensionale media*), which is the mean ascent speed, increases monotonically with the slope. Also, we show that VAM is maximized along a slope that is constant.

We begin this paper by presenting a model to account for the power required to maintain a given ground speed. Subsequently, we express VAM in terms of the ground speed and the slope. We proceed to prove that — for a given power — VAM increases monotonically with the slope, and that the maximization of VAM requires a constant slope. To gain an insight into these results, we examine a numerical example wherein we vary the values of cadence and gear ratio. We conclude by suggesting possible consequences of our results for VAM-maximization strategies.

2 Power

A standard mathematical model to account for the power required to propel a bicycle with speed V is (e.g., Danek et al., 2020)

$$P = F_{\leftarrow} V \tag{1}$$

$$= \frac{\underbrace{mg \sin \theta}_{\text{gravity}} + \underbrace{ma}_{\text{change of speed}} + \underbrace{C_{rr} mg \cos \theta}_{\text{rolling resistance}} + \underbrace{\frac{1}{2} \eta C_d A \rho (V + w_{\leftarrow})^2}_{\text{air resistance}}}{\underbrace{1 - \lambda}_{\text{drivetrain efficiency}}} V,$$

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where F_{\leftarrow} stands for the forces opposing the motion and V for the ground speed. Herein, m is the mass of the cyclist and the bicycle, g is the acceleration due to gravity, θ is the slope of a hill, a is the change of speed, C_{rr} is the rolling-resistance coefficient, $C_d A$ is the air-resistance coefficient, ρ is the air density, w_{\leftarrow} is the wind component opposing the motion, λ is the drivetrain-resistance coefficient, η is a quantity that ensures the proper sign for the tailwind effect, $w_{\leftarrow} < -V \iff \eta = -1$, otherwise, $\eta = 1$.

To consider a steady ride, $a = 0$, in windless conditions, $w = 0$, we write expression (1) as

$$P = \frac{mg \sin \theta + C_{rr} mg \cos \theta + \frac{1}{2} C_d A \rho V^2}{1 - \lambda} V. \quad (2)$$

3 VAM

Commonly, VAM is stated as a mean ascent speed in metres per hour. Since expressions (1) and (2) are commonly expressed in the *SI* units, let us define

$$\text{VAM} := 3600 V(\theta) \sin \theta. \quad (3)$$

As we proceed to prove, for a given value of P , VAM, is a monotonic function of θ ; hence, there is no maximum.

Setting the power to be $P = P_0$, and using expression (2), we write

$$V^3 + A(\theta) V + B = 0, \quad (4)$$

where

$$A(\theta) = \frac{mg}{\frac{1}{2} C_d A \rho} (\sin \theta + C_{rr} \cos \theta), \quad (5a)$$

$$B = -\frac{(1 - \lambda) P_0}{\frac{1}{2} C_d A \rho}, \quad (5b)$$

to state the following theorem.

Theorem 1. *For a constant power output, $P = P_0$, $V(\theta) \sin \theta$ increases monotonically for $0 \leq \theta \leq \pi/2$.*

Proof. Expressions (4), (5a) and (5b) determine $V = V(\theta)$. Then,

$$\frac{d}{d\theta} (V(\theta) \sin \theta) = V'(\theta) \sin \theta + V(\theta) \cos \theta, \quad (6)$$

where $'$ is a derivative with respect to θ . The derivative of expression (4) is

$$3V^2(\theta)V'(\theta) + A'(\theta)V(\theta) + A(\theta)V'(\theta) = 0,$$

from which it follows that

$$V'(\theta) = -\frac{A'(\theta)V(\theta)}{3V^2(\theta) + A(\theta)}. \quad (7)$$

Using expression (7) in expression (6), and simplifying, we obtain

$$\frac{d}{d\theta} (V(\theta) \sin \theta) = V(\theta) \frac{3V^2(\theta) \cos \theta - A'(\theta) \sin \theta + A(\theta) \cos \theta}{3V^2(\theta) + A(\theta)}. \quad (8)$$

Upon using expression (5a), the latter two terms in the numerator of expression (8) simplify to

$$\begin{aligned} -A'(\theta) \sin \theta + A(\theta) \cos \theta &= \frac{mg}{\frac{1}{2}C_d A \rho} \\ &\quad \left(-(\cos \theta - C_{rr} \sin \theta) \sin \theta \right. \\ &\quad \left. + (\sin \theta + C_{rr} \cos \theta) \cos \theta \right) \\ &= \frac{mg C_{rr}}{\frac{1}{2}C_d A \rho}. \end{aligned}$$

Hence,

$$\frac{d}{d\theta} (V(\theta) \sin \theta) = V(\theta) \frac{3V^2(\theta) \cos \theta + \frac{mg C_{rr}}{\frac{1}{2}C_d A \rho}}{3V^2(\theta) + A(\theta)} > 0, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad (9)$$

as required. Since expression (9) is positive, VAM is — within the range of interest — a monotonically increasing function of θ . \square

From this theorem, it follows that the steeper the incline that a rider can climb with a maximum power output, the greater the value of VAM; there is no θ that maximizes expression (3). Hence — based on expression (2), alone — we cannot specify the optimal slope to maximize VAM. To examine such a question, one needs to include other considerations.

For instance, we can consider

$$P = f v, \quad (10)$$

where f is the force applied to the pedals and v is the circumferential speed of the pedals. These are measurable quantities; they need not be modelled in terms of the surrounding conditions, such as the mass of the rider, strength and direction of the wind or the steepness of an uphill. To proceed, let us specify ℓ_c , which is the crank length, g_r , which is the gear ratio, r_w , which is the wheel radius, and c , which is the cadence, with all quantities expressed in the *SI* units. Hence, the circumferential speed is

$$v = 2\pi \ell_c c, \quad (11)$$

with the corresponding ground speed of

$$V = 2\pi r_w c g_r, \quad (12)$$

which allows us to examine VAM as a function of the cadence and gear ratio.

4 Brachistochrone

Prior to examining VAM as a function of the cadence and gear ratio, let us return to expressions (4), (5a) and (5b) to consider the brachistochrone problem, which herein consists of finding the curve along which the ascent between two points — under the assumption of constant power — takes the least amount of time.

Theorem 2. *The path of quickest ascent from $(0,0)$ to (R,H) , which is an elevation gain of H over a horizontal distance of R , is the straight line, $y = (H/R)x$, where $0 \leq x \leq R$ and $y(R) = H$.*

Proof. The relation between speed, slope and power is stated by expressions (4), (5a) and (5b), which determine $V = V(\theta)$, where θ is the slope, namely,

$$\tan \theta = \frac{dy}{dx}, \quad \theta = \arctan \left(\frac{dy}{dx} \right).$$

The traveltime over an infinitesimal distance, ds , is $dt = ds/V$, where V is the ground speed. Since $ds^2 = dx^2 + dy^2$, we write

$$dt = \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}{V(\theta)} = \frac{\sqrt{1 + (y')^2}}{V(\arctan(y'))} dx,$$

where $y' := dy/dx$. Hence, the ascent time is

$$T = \int_0^R \frac{\sqrt{1 + (y')^2}}{V(\arctan(y'))} dx =: \int_0^R \underbrace{\frac{\sqrt{1 + p^2}}{V(\arctan(p))}}_F dx,$$

where $p := y'(x)$. Thus, we write the integrand as a function of three independent variables,

$$T =: \int_0^R F(x, y, p) dx.$$

To minimize the traveltime, we invoke the Euler-Lagrange equation,

$$\frac{\partial}{\partial y} F(x, y, p) = \frac{d}{dx} \frac{\partial}{\partial p} F(x, y, p).$$

Since F is not an explicit function of y , the left-hand side is zero,

$$0 = \frac{d}{dx} \left\{ \frac{\partial}{\partial p} \frac{\sqrt{1 + p^2}}{V(\arctan(p))} \right\}, \quad (13)$$

which means that the term in braces is a constant. To proceed, we use the fact that

$$p = \tan \theta \implies \frac{\partial}{\partial p} = \frac{d\theta}{dp} \frac{\partial}{\partial \theta} = \cos^2 \theta \frac{\partial}{\partial \theta}$$

and

$$\begin{aligned} \frac{\partial}{\partial p} \frac{\sqrt{1 + p^2}}{V(\arctan(p))} &= \cos^2 \theta \frac{\partial}{\partial \theta} \frac{\sqrt{1 + p^2}}{V(\arctan(p))} = \cos^2 \theta \frac{\partial}{\partial \theta} \frac{\sqrt{1 + \tan^2 \theta}}{V(\theta)} = \cos^2 \theta \frac{\partial}{\partial \theta} \frac{\sqrt{\sec \theta}}{V(\theta)} \\ &= \cos^2 \theta \frac{d}{d\theta} \frac{1}{\cos \theta V(\theta)}, \end{aligned}$$

for $0 \leq \theta \leq \pi/2$. Thus, expression (13) is

$$\frac{d}{dx} \left(\cos^2 \theta \frac{d}{d\theta} \frac{1}{\cos \theta V(\theta)} \right) \equiv 0,$$

which, using the fact that

$$\theta = \arctan(y'(x)) \implies \frac{d\theta}{dx} = \frac{1}{1 + (y'(x))^2} y''(x) \quad \text{and} \quad \frac{d}{dx} = \frac{d\theta}{dx} \frac{d}{d\theta},$$

we write as

$$\frac{y''(x)}{1 + (y'(x))^2} \frac{d}{d\theta} \left(\cos^2 \theta \frac{d}{d\theta} \frac{1}{\cos \theta V(\theta)} \right) \equiv 0.$$

Hence, there are two possibilities. Either

$$\underbrace{y''(x) \equiv 0}_{(a)} \quad \text{or} \quad \underbrace{\frac{d}{d\theta} \left(\cos^2 \theta \frac{d}{d\theta} \frac{1}{\cos \theta V(\theta)} \right) \equiv 0}_{(b)}.$$

We claim that (b) is impossible. Indeed, (b) holds if and only if

$$\cos^2 \theta \frac{d}{d\theta} \frac{1}{\cos \theta V(\theta)} \equiv a,$$

where a is a constant. It follows that

$$\frac{d}{d\theta} \frac{1}{\cos \theta V(\theta)} = a \sec^2 \theta \iff \frac{1}{\cos \theta V(\theta)} = a \tan \theta + b,$$

where a and b are constants, and

$$\cos \theta V(\theta) = \frac{1}{a \tan \theta + b} \iff V(\theta) = \frac{1}{a \sin \theta + b \cos \theta}.$$

This cannot be the case, since it implies that there exists a value of θ^* for which $a \sin \theta^* + b \cos \theta^* = 0$ and $V(\theta^*) \rightarrow \infty$. But $V^3 + A(\theta)V + B = 0$, so any root of this cubic equation is such that

$$|V| \leq \max\{1, |A(\theta)| + |B|\}.$$

Thus, it follows that (a), namely, $y''(x) \equiv 0$, must hold, which implies that $y(x)$ is a straight line. For the ascent from $(0, 0)$ to (R, H) , the line is $y(x) = (H/R)x$, as required. \square

In the context of VAM, there is a corollary of Theorem 2.

Corollary 1. *The path of the quickest gain of altitude between two points, $(0, 0)$ and (R, H) , is the straight line, $y = (H/R)x$, where $0 \leq x \leq R$ and $y(R) = H$.*

Let us emphasize that this corollary, as well as Theorem 2, are statements of mathematical physics, whose consequences might be adjusted by other factors to propose an actual strategy. For instance, it might be preferable — from a physiological viewpoint — to vary the slope in order to allow a rider periods of respite.

5 Numerical example

5.1 Formulation

To examining VAM as a function of the cadence and gear ratio, let us consider the following values. For the bicycle-cyclist system, $m = 111$, $C_{dA} = 0.2702$, $C_{rr} = 0.01298$, $\lambda = 0.02979$. For the bicycle, $\ell_c = 0.175$, $g_r = 1.5$ and $r_w = 0.335$. For the external conditions, $g = 9.81$ and $\rho = 1.16826$.

Let us suppose that the power output that the rider keeps during a climb is $P_0 = 375$ and that the cadence is $c = 1$, which — in accordance with expression (11) — results in $v = 1.0996$. The force that the rider must apply to the pedals is $f = 341.0463$. In accordance with expression (12), the ground speed is $V = 3.1573$.

Inserting this value of V into expression (4), we obtain

$$45.9957 \cos \theta + 3543.5790 \sin \theta - 369.8799 = 0,$$

whose solution is $\theta = 0.0916$, which results in $\text{VAM} = 1039.5$; herein, θ corresponds to a grade of 9.1840%.

5.2 VAM as a function of cadence and gear ratio

Let us consider the effect on the value of VAM due to varying the cadence and gear ratio. To do so, we use expression (12) in expressions (3) and (4) to obtain

$$\text{VAM} = 7577.5215 c g_r \sin \theta \tag{14}$$

and

$$1.5171 (c g_r)^3 + 30.6638 c g_r \cos \theta + 2362.3863 c g_r \sin \theta - 375 = 0, \tag{15}$$

respectively. In both expressions (14) and (15), c and g_r appear as a product.

For each $c g_r$ product, we solve equation (15) to obtain the corresponding value of θ , which we use in expression (14) to calculate the VAM. We consider $c g_r \in [0.5, 3.75]$; the lower limit can represent $c = 0.5$, which is 30 rpm, and $g_r = 1$; the upper limit can represent $c = 1.5$, which is 90 rpm, and $g_r = 2.5$. Following expression (11), the corresponding circumferential speeds are $v = 0.55$ and $v = 1.65$. Since $P_0 = 375$, in accordance with expression (10), the forces applied to the pedals are $f = 682$ and $f = 227$, respectively.

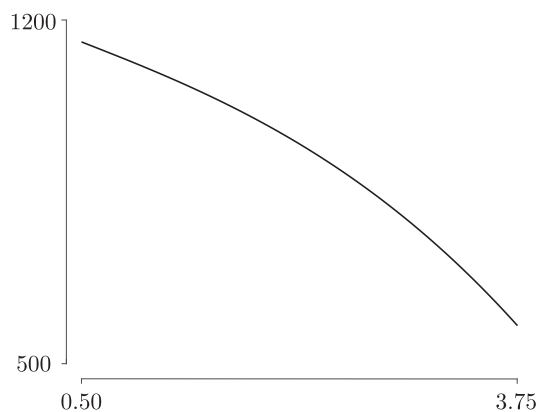


Figure 1: VAM as function of $c g_r$, with $P_0 = 375$

Examining Figure 1, we conclude that lowering $c g_r$ increases the slope of the climbable incline and, hence—for a constant value of power—increases the VAM, as expected in view of Theorem 1.

The VAM is determined not only by the power output sustainable during a climb, but also the steepness of that climb. As stated in Theorem 1, the steeper the slope that can be climbed with a given power, the greater the VAM. Hence, as illustrated in Figure 1, the highest value corresponds to the lowest cadence-gear product. A maximization of VAM requires an optimization of the force applied to the pedals and their circumferential speed, along a slope, in order to maintain a high power.

6 Conclusions

The presented results constitute a mathematical-physics background upon which various strategies for the VAM maximization can be examined. In particular, in the context of physiological considerations, one could examine the maximum sustainable power as a function of both the force applied to the pedals and their circumferential speed. For instance, one might choose a less steep slope to allow a higher $c g_r$ product, whose value allows to generate and sustain a higher power. In all cases, however, the slope needs to be constant.

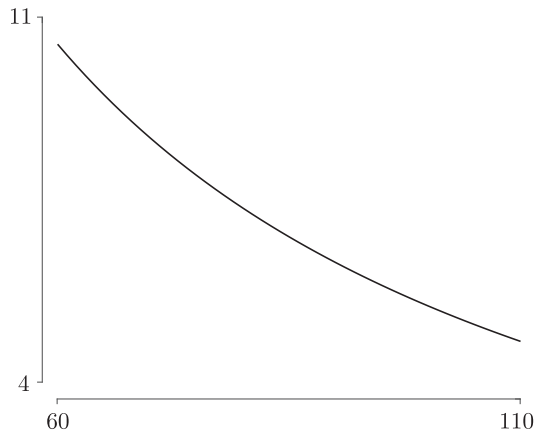


Figure 2: Slope, in %, as function of m , with $P_0 = 375$ and $c g_r = 2.25$

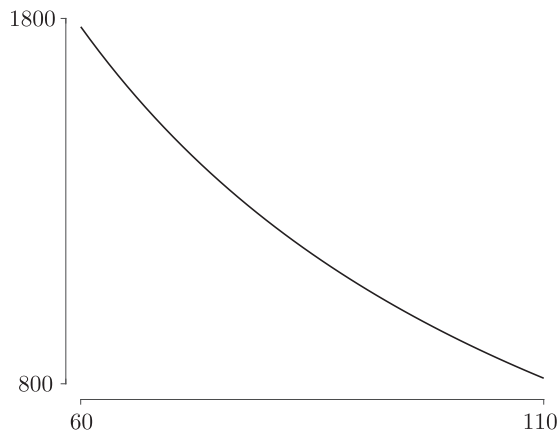


Figure 3: VAM as function of m , with $P_0 = 375$ and $c g_r = 2.25$

Let us revisit expression (14), whose general form, in terms of $c g_r$, is

$$8 \pi^3 r_w^3 (c g_r)^3 + \frac{2 \pi r_w m g (\sin \theta + C_{rr} \cos \theta)}{\frac{1}{2} C_d A \rho} c g_r - \frac{P_0 (1 - \lambda)}{\frac{1}{2} C_d A \rho} = 0.$$

Using common values for most quantities, namely,

$$C_d A = 0.27, \quad C_{rr} = 0.02, \quad \lambda = 0.03, \quad \rho = 1.2, \quad r_w = 0.335,$$

and, for convenience of a concise expression, below, we invoke the small-angle approximation, which results—in radians—in $\sin \theta \approx \theta$ and $\cos \theta \approx 1 - \theta^2/2$, to obtain

$$\theta = 50 - 15.5610 \sqrt{10.3326 + \frac{0.0302 c g_r - 0.0194 P_0}{m c g_r}},$$

where m is the mass of the bicycle-cyclist system, and the product, $c g_r$, might be chosen to correspond to the maximum sustainable power, P_0 . This product is related to power by expression (10), since c is proportional to v , and g_r to f . For any value of $c g_r$, there is a unique value of P_0 , which depends only on the output of a rider.

As expected, and as illustrated in Figure 2, the slope decreases with m , *ceteris paribus*. Also, as expected, and as illustrated in Figure 3, so does VAM. For both figures, the abscissa, expressed in kilograms, can be viewed as the power-to-weight ratio from 6.25 to 3.41, whose limits represent the values sustainable—for about an hour—by an elite and a moderate rider, respectively.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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